NUMERICAL METHODS FOR COMPUTING THE DIFFUSION OF DISTURBANCES ON SONIC BOOM WAVES

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I. INTRODUCTION

This report describes the work performed during the first six-month period under Grant Number NGR-37-002-037 of the National Aeronautics and Space Administration. The general objective of this project is to develop finite-difference methods for the computation of sonic boom wave reflections and diffractions.

The numerical method proposed was an explicit Eulerian solution of the basic fluid flow conservation equations, applied to a field of discrete computation points. Since such equations are non-linear partial differential equations of the hyperbolic type, they are numerically unstable when computed in an explicit, march-forward-in-time method. Stability is provided by the addition of dissipative diffusion terms which are everywhere small except in regions of large spatial second derivatives. Thus, the shock wave discontinuity, which is the principal destabilizing disturbance, is smoothed into a thin region having high gradients.

NOMENCLATURE

a	Acoustic velocity
e	Energy per unit volume
p	Pressure
t	Time
u	Velocity component in the x-direction
v	Velocity component in the y-direction
V	Velocity magnitude
w	Velocity component in the z-direction
x,y,z	Cartesian coordinates
γ	Ratio of specific heats
ρ	Density
ξ	Pressure ratio across a shock wave

II. ATTEMPTS TO SOLVE THE WEAK-WAVE EQUATION

For weak pressure waves, including sonic booms at far field conditions, the simple pressure wave equations applies:

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} = \frac{1}{a^2}\frac{\partial^2 p}{\partial t^2} \quad ,$$

where p is the pressure perturbation.

This was attractive for numerical solutions because it involved only one dependent variable. However, two difficulties were apparent at the start: (1) second-order derivatives were involved, making explicit determination difficult; (2) the boundary representation was troublesome at solid walls. An incoming wave had to be so represented that a proper reflected wave would result. No method could be devised to reflect waves in a physically correct manner.

It was therefore necessary to turn to the complete set of equations for fluid motion, rather than merely solving the acoustic equation.

III. THE CONSERVATION EQUATIONS FOR WEAK-WAVE SYSTEMS

The equations for conservation of mass, of momentum in the three space dimensions, and of energy are, respectively:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0 \tag{1}$$

$$\frac{\partial \rho u}{\partial t} + \frac{\partial}{\partial x}(\rho u^2 + p) + \frac{\partial \rho uv}{\partial y} + \frac{\partial \rho uw}{\partial z} = 0 \tag{2}$$

$$\frac{\partial \rho v}{\partial t} + \frac{\partial \rho uv}{\partial x} + \frac{\partial}{\partial y}(\rho v^2 + p) + \frac{\partial \rho vw}{\partial z} = 0 \qquad (3)$$

$$\frac{\partial \rho w}{\partial t} + \frac{\partial \rho uw}{\partial x} + \frac{\partial \rho vw}{\partial y} + \frac{\partial}{\partial z}(\rho w^2 + p) = 0 \qquad (4)$$

$$\frac{\partial e}{\partial t} + \frac{\partial (e+p)u}{\partial x} + \frac{\partial (e+p)v}{\partial y} + \frac{\partial (e+p)w}{\partial z} = 0 \qquad (5)$$

Here, e is the energy per unit volume and

$$e = \frac{1}{2}\rho(u^2 + v^2 + w^2) + \frac{p}{\gamma - 1} \qquad (6)$$

describes the sum of the kinetic and internal energies. When (2), (3) and (4) are expanded and combined with (1), they become

$$\frac{\partial u}{\partial t} + \frac{1}{\rho}\frac{\partial p}{\partial x} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = 0 \qquad (7)$$

$$\frac{\partial v}{\partial t} + \frac{1}{\rho}\frac{\partial p}{\partial y} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = 0 \qquad (8)$$

$$\frac{\partial w}{\partial t} + \frac{1}{\rho}\frac{\partial p}{\partial z} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = 0 \qquad (9)$$

Equations (1), (6), (7), (8), and (9) can be so combined with (5) that it reduces to

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z} + \gamma p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0 \qquad (10)$$

For the sake of brevity, let differentiation be denoted by the subscript. The equations are then written as follows:

$$u_t + \frac{1}{\rho}p_x + u\,u_x + vu_y + wu_z = 0 \qquad (7)$$

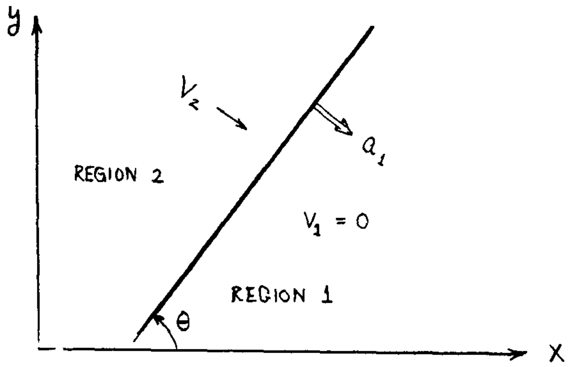
$$v_t + \frac{1}{\rho}p_y + uv_x + vv_y + wy_z = 0 \qquad (8)$$

$$w_t + \frac{1}{\rho} p_z + uw_x + vw_y + ww_z = 0 \qquad (9)$$

$$p_t + up_x + vp_y + wp_z + \gamma p(u_x + v_y + w_z) = 0 \qquad (10)$$

A. Reduced Form for a Moving Weak Wave

To simplify the equations for weak wave conditions, an order-of-magnitude estimate was made. Typical sonic boom values are listed below.



a_1 = 1100 ft/sec.

p_1 = 2000 lb/ft²

ρ_1 = .0023 slugs/ft³

p_2 = 2002 lb./ft²

ΔP = 2 lb/ft²

$\xi = p_2/p_1 = 1.001$; and for $(\xi - 1) \ll 1$,

$$V_2 = a_1 \frac{2(\xi - 1)}{\sqrt{2\gamma\left[\gamma + 1)\xi + (\gamma - 1)\right]}} = 0.393 \text{ ft/sec}^2$$

$$\rho_2 = \rho_1\left[\frac{(\gamma + 1)\xi + (\gamma - 1)}{(\gamma - 1)\xi + (\gamma + 1)}\right] = \rho_1 \left(\frac{2.8024}{2.8004}\right) \cong \rho_1$$

If $\theta = 45^\circ$

$$u_2 = v_2 = 0.281 \text{ ft/sec.}$$

If $a_1 dt \cong dx = dy = dz = 1$, then each term of (7), (8) and (9) is seen to be of order 1000 or 1, and each term of (10) to be of order 10^{-3} or 10^{-6}. Retaining only the larger terms reduces these form equations to

$$u_t + \frac{1}{\rho} p_x = 0 \qquad (11)$$

$$v_t + \frac{1}{\rho} p_y = 0 \qquad (12)$$

$$w_t + \frac{1}{\rho} p_z = 0 \qquad (13)$$

$$p_t + \gamma p(u_x + v_y + w_z) = 0. \qquad (14)$$

B. Form for a Stationary, Weak Shock

It is often convenient to shift the coordinate system to the wave. When this is done, the order-of-magnitude analysis does not justify any simplification and equations (7) through (10) must be used unaltered.

C. Finite Difference Form of the Equations

Equation (11) becomes, in finite difference form,

$$u_{k,\ell}^{n+1} = u_{k,\ell}^{n} - \frac{\Delta t}{2\rho\,\Delta x}\left(P_{k+1,\ell,m} - P_{k-1,\ell,m}\right)^{n} \qquad (11)$$

Here k is the point location in the x-direction, ℓ in the y-direction, and m in z-direction. The time plane is n and the time and space increments are Δt and Δx. The other equations can be similarly represented.

Solid boundaries may be represented by a "reflection point" one increment inside the wall having properties identical to those at one increment outside, except that the velocities normal to the wall are in opposite directions.

These equations, as written above, are numerically unstable. They can be stabilized by the addition of diffusion terms, as is done in many numerical field-plotting methods.

IV. COMPARISON WITH ANALYTICAL RESULTS

To evaluate the numerical method presented above, several test problems were programmed and computed on the IBM 7040 computer at the Oklahoma State University Computer Center.

A set of simple plane wave/wall interaction problems were run, and the results were physically perceptive. As a more rigid test, a two-dimensional problem was computed involving an "N-wave" impinging on the area between the ground and a finite-length overhanging canopy. This corresponded exactly to the problem solved by an analytical method in the work performed by the project director with Andrews Associates under a recent N.A.S.A. contract. This was described in detail in N.A.S.A. Contractor Report 66169.

The results of the two methods were nearly identical except that the numerical method tended to smooth the sharpest "spikes" on the pressure-time curves. This smoothing is the inevitable effect of the addition of the diffusion terms for numerical stability, and of using a finite number of computation net points. However, the effect was not great enough to cause serious errors. Since the analytical method rests on firm physical and mathematical foundations, the agreement was accepted as establishing a good level of confidence in the results of the numerical technique.

V. APPLICATION TO PLANE-WAVE ATMOSPHERIC DISTURBANCES

The weak-wave numerical method is now being applied to a plane wave moving through an atmosphere which is uniform except for a thin, horizontal gust which is exerted for a short period of time. The extent of the disturbance on the moving wave will be computed. When this is computed, the uniform atmosphere will be replaced by one having a linear variation of acoustic velocity with altitude, and horizontal and vertical gusts will be imposed.

Problems planned, in order to increasing complexity, are:

(a) plane wave with local plane temperature anamoly.

(b) plane wave with local (circular) gusts and temperature anamolies. This becomes an axi-symmetric problem for a uniform atmosphere and a three-dimension problem for the variable temperature atmosphere.

VI. THREE-DIMENSIONAL STRONG WAVE METHOD

Finite difference forms of Equations (1) through (5) have been written in cylindrical coordinates, including the artifical diffusion terms. These were given an initial evaluation by application to a cone-cylinder at a 5° angle of attack flying at a Mach number of 3.0. The computations were performed at 400 points for each of ten radial planes passing through the axis of symmetry. The resulting flow fields are being compared with those from analytical methods (Kopal's and Sims' tables). When the development of the computer code is complete, it will be possible to apply it to the near-field computations of a non-symmetrical body, such as half-cone.

The three-dimensional program is quite lengthy and time-consuming, due to the large number of points required to describe the field. The 7040 computer can be utilized only by a method of constant shifting of stored information between the computer and magnetic tape units. This is very slow and inefficient use of a computer. It may be necessary to appeal to the sponsoring agency to provide a large capacity computer for truly meaningful production runs.